# Renormalization-group approach to the dynamical Casimir effect

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In this paper we study the one-dimensional dynamical Casimir effect. We consider a one-dimensional cavity formed by two mirrors, one of which performs an oscillatory motion with a frequency resonant with the cavity. The naive solution, perturbative in powers of the amplitude, contains secular terms. Therefore it is valid only in the short time limit. Using a renormalization-group technique to resum these terms, we obtain an improved analytical solution which is valid for longer times. We discuss the generation of peaks in the density energy profile and show that the total energy inside the cavity increases exponentially. [S1050-2947(98)03103-5]

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## I. INTRODUCTION

The problem of quantum fluctuations inside cavities has attracted attention for many years [1]. A way to study the structure of the vacuum is to distort it by changing the configuration of the cavity in time [2,3]. The simplest configuration is that of a one-dimensional cavity formed by two perfectly reflecting mirrors, one of which is fixed and the other one is allowed to move in a predetermined way, or rather, its motion is determined by the backreaction of the electromagnetic field [4]. There are in fact only a few predetermined motions which allow an exact resolution of the problem. In [5] a special motion for the mirror, which has an exact solution, has been considered, and it has been shown that the Casimir force may be resonantly enhanced. In [6] a geometrical method for solving the problem for arbitrary wall motions has been developed, and basically the same structure for the electromagnetic field within the cavity has been found. Of special interest are the cases where the moving mirror oscillates with one of the eigenfrequencies of the unperturbed cavity [7–9]. A naive approach is to make perturbations in the amplitude of oscillation. However, this perturbative treatment has only a very limited range of validity: the appearance of secular terms proportional to the time implies that after a short period the approximation breaks down. In this paper we apply a method inspired in the renormalization group (RG) to treat these singular perturbations. The method has a wide range of application [10], especially to ordinary differential equation problems involving multiple scales, boundary layers, asymptotic matching, and WKB analysis. The main advantage of the RG method is to provide a simple and unified calculational method for all problems of this sort. In our present case, the application of the method to the dynamical Casimir effect permits us to get a solution for the structure of the electromagnetic field within the cavities that is valid for a period of time longer than that of the perturbative case. With the solution at hand, we study local properties of the field such as the energy density. In agreement with other authors [6], we show that the resonant moving wall induces an exponential growth of the total energy, and that peaks form inside the cavity, which travel at the speed of light bouncing against the walls.

We consider a one-dimensional cavity formed by two perfectly reflecting mirrors. One of them is fixed at $x=0$, while the other one performs an oscillatory motion $L(t)=L_0[1+\epsilon\sin(q\pi t/L_0)]$, with $\epsilon\ll 1$ and $q\in\mathbb{N}$, i.e., the moving mirror oscillates with a frequency equal to one of the eigenfrequencies of the cavity. We shall assume that the oscillations begin at $t=0$, and that the mirror is at rest for $t<0$. Note that we shall not treat the moving mirror as a degree of freedom (either classical or quantum), but just as a given time-dependent boundary for the electromagnetic field inside the cavity. The vector potential $A(x,t)$ satisfies the one-dimensional field equation $\Box A=0$ and the boundary conditions $A(x=0,t)=A(L(t),t)=0$ for all times. Therefore one can express the field inside the cavity as

$$A(x,t)=\sum_{k=1}^{\infty}[a_k\psi_k(x,t)+a_k^{\dagger}\psi_k^{*}(x,t)], \tag{1}$$

where the mode functions $\psi_k(x,t)$ are positive frequency modes for $t<0$, and $a_k$ and $a_k^{\dagger}$ are time-independent annihilation and creation operators, respectively.

If one writes the modes in terms of a function $R(t)$ as [2]

$$\psi_k(x,t)=\frac{i}{\sqrt{4\pi k}}\left(e^{-ik\pi R(t+x)}-e^{-ik\pi R(t-x)}\right), \tag{2}$$

the boundary conditions are met provided that

$$R(t+L(t))-R(t-L(t))=2. \tag{3}$$

The complete solution to the problem involves finding a solution $R(t)$ in terms of the prescribed motion $L(t)$. The modes are positive frequency modes for $t<0$ if $R(t)=t/L_0$ for $-L_0\leqslant t\leqslant L_0$, which is indeed a solution to Eq. (3) for $t<0$. Note that the boundary condition for $R(t)$ involves its values over the whole range of time $-L_0\leqslant t\leqslant L_0$.

In what follows, we will describe a method of finding an analytical approximation to the solution of Eq. (3). We shall first obtain a perturbative solution by expanding in powers of the amplitude $\epsilon$ (Sec. III). As this perturbative solution will

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contain secular terms, it will be valid only for short times, i.e., $\epsilon t/L_0 < 1$. Using RG techniques, we will be able to perform a resummation of the secular terms and obtain an analytical approximation valid for a longer period of time $\epsilon^2 t/L_0 < 1$ (Sec. IV). In Sec. V we will use this solution to describe the evolution of the mean value of the energy density inside the cavity. In order to get acquainted with the renormalization-group method, in Sec. II we will describe a simple example where the resummation of the secular terms is performed for a particular ordinary differential equation. The reader already familiar with the work of Ref. [10] can skip the next section.

## II. A SIMPLE EXAMPLE

Let us consider the Rayleigh equation

$$\frac{d^2y}{dt^2} + y + \epsilon\left\{\frac{1}{3}\left(\frac{dy}{dt}\right)^3 - \frac{dy}{dt}\right\} = 0, \tag{4}$$

where $\epsilon$ is a small number. This is an interesting oscillator because it can be shown that, for any initial condition and any positive $\epsilon$, the exact solution becomes periodic at long times and therefore approaches a limit circle in phase space [11].

The Rayleigh equation can be solved perturbatively using an expansion in powers of $\epsilon$, that is, $y = y_0 + \epsilon y_1 + O(\epsilon^2)$. Up to first order in $\epsilon$, the perturbative solution reads

$$\begin{aligned} y(t) = {} & Y_0 \sin(t+\Theta_0) + \epsilon\left\{\frac{Y_0}{2}\left(1 - \frac{Y_0^2}{4}\right)(t-t_0)\sin(t+\Theta_0)\right. \\ & \left. + \frac{Y_0^3}{96}\{\cos[3(t+\Theta_0)] - \cos(t+\Theta_0)\}\right\} + O(\epsilon^2), \end{aligned} \tag{5}$$

where $Y_0$ and $\Theta_0$ are constants determined by the initial conditions at arbitrary $t = t_0$. This perturbative solution does not become periodic and, therefore, it is not a good approximation for long times. Indeed, due to the presence of the secular term, the naive perturbative solution is valid only for times close to the initial time $t_0$, and breaks down for $\epsilon\,(t - t_0) \geqslant 1$. This is typical of systems showing parametric resonance. Usually one has a system weakly coupled to an external resonant force, and as one tries to make a perturbative analysis, the corrections possess secular terms, i.e., terms that grow linearly with time. In the following we shall adopt the RG method to treat singular perturbations and we shall use it to make an improvement to the perturbative solution, which shall be valid for longer times, $\epsilon^2(t-t_0) < 1$. The basic idea [10] is to introduce an arbitrary time $\tau$, split $t - t_0$ as $(t-\tau) + \tau - t_0$, and absorb the terms proportional to $\tau - t_0$ into the ‘‘renormalized’’ counterparts $Y(\tau)$ and $\Theta(\tau)$ of the ‘‘bare’’ parameters contained in the zeroth order solution, that is, $Y_0$ and $\Theta_0$. Using this idea one eliminates the secular terms proportional to $\tau - t_0$, and the function $y(t)$ takes the form

$$\begin{aligned} y(t) = {} & Y\sin(t+\Theta) + \epsilon\left\{\frac{Y}{2}\left(1 - \frac{Y}{4}\right)(t-\tau)\sin(t+\Theta)\right. \\ & \left. + \frac{Y^3}{96}\{\cos[3(t+\Theta)] - \cos(t+\Theta)\}\right\} + O(\epsilon^2), \end{aligned} \tag{6}$$

where now $Y$ and $\Theta$ are functions of $\tau$. Now comes the crucial point. As $\tau$ does not appear in the original equation or in the initial conditions, the solution Eq. (6) should not depend on $\tau$. Therefore the partial derivative with respect to $\tau$ should vanish, i.e., $(\partial y/\partial\tau)_t = 0$ for any $t$. This is the RG equation, which implies

$$\frac{dY}{d\tau} = \epsilon\frac{Y}{2}\left(1 - \frac{Y^2}{4}\right) + O(\epsilon^2),$$

$$\frac{d\Theta}{d\tau} = O(\epsilon^2). \tag{7}$$

The solutions to these equations are

$$\begin{aligned} Y(\tau) = {} & Y(t_0)\left[e^{-\epsilon(\tau-t_0)} + \frac{Y^2(t_0)}{4}(1 - e^{-\epsilon(\tau-t_0)})\right]^{-1/2} \\ & + O(\epsilon^2(\tau-t_0)), \end{aligned}$$

$$\Theta(\tau) = \Theta(t_0) + O(\epsilon^2(\tau-t_0)), \tag{8}$$

where $Y(t_0)$ and $\Theta(t_0)$ are constants to be determined by the initial conditions. We still have the freedom to choose the arbitrary time $\tau$. The obvious choice is $t = \tau$, since in this way the secular term proportional to $t - \tau$ in Eq. (6) disappears. Assuming the initial condition $y(t_0) = 0$, $\dot{y}(t_0) = 2a$, with $a$ any real number, we find $Y(t_0) = 2a$ and $\Theta(t_0) = -t_0$. Finally, the RG-improved solution reads

$$\begin{aligned} y(t) = {} & Y(t)\sin(t-t_0) + \epsilon\frac{Y(t)^3}{96}\{\cos[3(t-t_0)] - \cos(t-t_0)\} \\ & + O(\epsilon^2), \end{aligned} \tag{9}$$

that is valid for $\epsilon^2(t-t_0) < 1$. Note that the improved solution becomes periodic and approaches a limit circle of radius 2 for $\epsilon(t-t_0) \geqslant 1$.

It is interesting to remark the analogy with the usual RG approach in quantum field theory: $t_0$ plays the role of the ultraviolet cutoff (although there are no divergences but secular terms here), $Y_0$ and $\Theta_0$ are the bare coupling constants, and $Y$ and $\Theta$ are the renormalized counterparts evaluated at the ‘‘scale’’ $\tau$. The equation is ‘‘renormalizable’’ because the secular terms can be absorbed into the bare parameters. As anticipated, the RG is a straightforward method by which to obtain, from the naive perturbative solution, an improved solution which is valid for longer times. In this particular example, it can be shown that the RG method is equivalent to multiple-scale analysis [11], with the additional practical advantage that it is not necessary to know *a priori* the multiple time scales.

## III. PERTURBATIVE SOLUTION TO THE DYNAMICAL CASIMIR EFFECT

We will now solve Eq. (3) using a naive perturbative expansion. We expand the function $R(t)$ in terms of the small amplitude $\epsilon$ and retain first order terms only, $R(t) = R_0(t) + \epsilon R_1(t)$. Equating terms of the same order we get

$$R_0(t+L_0) - R_0(t-L_0) = 2, \tag{10}$$

$$R_1(t+L_0) - R_1(t-L_0) = -L_0 \sin\left(\frac{q\pi t}{L_0}\right)[R_0'(t+L_0) + R_0'(t-L_0)]. \tag{11}$$

The general solution to Eq. (10) is

$$R_0(t) = a + \frac{t}{L_0} + \sum_{n\geqslant 1}\left[A_n \sin\left(\frac{n\pi t}{L_0}\right) + B_n \cos\left(\frac{n\pi t}{L_0}\right)\right], \tag{12}$$

where $a, A_n$, and $B_n$ are constants determined by the boundary condition, that is, by the value of $R(t)$ for $-L_0 \leqslant t \leqslant L_0$. Introducing this solution into Eq. (11) we obtain

$$\begin{aligned}-\frac{1}{2}&[R_1(t+L_0) - R_1(t-L_0)] \\ &= \sin\left(\frac{q\pi t}{L_0}\right) + \frac{\pi}{2}\sum_{n\geqslant 1} n(-1)^n \left\{A_n\left[\sin\left(\frac{(q+n)\pi t}{L_0}\right) + \sin\left(\frac{(q-n)\pi t}{L_0}\right)\right] + B_n\left[\cos\left(\frac{(q+n)\pi t}{L_0}\right) - \cos\left(\frac{(q-n)\pi t}{L_0}\right)\right]\right\},\end{aligned} \tag{13}$$

whose general solution reads

$$\begin{aligned}R_1(t) = (-1)^{q+1}\frac{t}{L_0}\Bigg(&\sin\left(\frac{q\pi t}{L_0}\right) + \frac{\pi}{2}\sum_{n\geqslant 1} n\left\{A_n\left[\sin\left(\frac{(q+n)\pi t}{L_0}\right) + \sin\left(\frac{(q-n)\pi t}{L_0}\right)\right]\right. \\ &\left.+ B_n\left[\cos\left(\frac{(q+n)\pi t}{L_0}\right) - \cos\left(\frac{(q-n)\pi t}{L_0}\right)\right]\right\}\Bigg) + g(t),\end{aligned} \tag{14}$$

where $g(t)$ is an arbitrary periodic function with period $2L_0$. We see that, as in the case of the Rayleigh oscillator, the perturbative correction contains secular terms that grow linearly in time. Therefore this approximation will be valid only for short times, that is, $\epsilon\, t/L_0 < 1$.

If we assume that the boundary condition for $R(t)$ is already satisfied by $R_0(t)$, then the periodic function $g(t)$ must be chosen in such a way that $R_1(t) = 0$ for $-L_0 \leqslant t \leqslant L_0$. Therefore

$$\begin{aligned}g(2pL_0 + z) = (-1)^q \frac{z}{L_0}\Bigg\{&\sin\left(\frac{q\pi z}{L_0}\right) + \frac{\pi}{2}\sum_{n\geqslant 1} n\left\{A_n\left[\sin\left(\frac{(q+n)\pi z}{L_0}\right)\right.\right. \\ &\left.+ \sin\left(\frac{(q-n)\pi z}{L_0}\right)\right] + B_n\left[\cos\left(\frac{(q+n)\pi z}{L_0}\right) \left.\left. - \cos\left(\frac{(q-n)\pi z}{L_0}\right)\right]\right\}\Bigg\},\end{aligned} \tag{15}$$

where $t = 2pL_0 + z$, $p = 0,1,2,\ldots$, and $-L_0 \leqslant z \leqslant L_0$. Given $t$, the value of the integer $p$ is obtained as $p = \frac{1}{2}\mathrm{int}(t/L_0)$ or $p = \frac{1}{2}[\mathrm{int}(t/L_0) + 1]$ for $\mathrm{int}(t/L_0)$ even or odd, respectively. Note that during the first period ($p = 0$), $g(t)$ makes $R_1(t)$ vanish identically. As we have already seen, since the mirror is at rest for $t < 0$, we must impose $R(t) = t/L_0$ for $-L_0 \leqslant t \leqslant L_0$. Therefore $a = A_n = B_n = 0$, and the perturbative solution reads

$$R(t) = \frac{t}{L_0} + \epsilon(-1)^{q+1}\left[\frac{t}{L_0}\sin\left(\frac{q\pi t}{L_0}\right) - \frac{z}{L_0}\sin\left(\frac{q\pi z}{L_0}\right)\right]. \tag{16}$$

The naive perturbative solution to the dynamical Casimir effect has been previously discussed in Ref. [7]. In that work the periodic function $g(t)$ was taken equal to zero, which resulted in the omission of the third term in Eq. (16). Thus the solution obtained there does not satisfy the correct boundary condition. Note, however, that after many periods ($1 \ll t/L_0 \ll \epsilon^{-1}$) both solutions practically coincide.

## IV. RENORMALIZATION-GROUP IMPROVEMENT

We will now adapt the RG method of Sec. II in order to obtain a solution to Eq. (3) which is valid beyond the short time limit. Let us introduce the arbitrary time $\tau$ and split $t$ as $t - \tau + \tau$. The perturbative solution can then be written as [see Eqs. (12) and (14)]

$$\begin{aligned}R(t) = a(\tau) &+ \sum_{n\geqslant 1}\left[A_n(\tau)\sin\left(\frac{n\pi t}{L_0}\right) + B_n(\tau)\cos\left(\frac{n\pi t}{L_0}\right)\right] \\ &+ \frac{t-\tau}{L_0} + \epsilon\frac{t-\tau}{L_0}(-1)^{q+1}\Bigg(\sin\left(\frac{q\pi t}{L_0}\right) \\ &+ \frac{\pi}{2}\sum_{n\geqslant 1} n(-1)^n\left\{A_n(\tau)\left[\sin\left(\frac{(q+n)\pi t}{L_0}\right) + \sin\left(\frac{(q-n)\pi t}{L_0}\right)\right] + B_n(\tau)\left[\cos\left(\frac{(q+n)\pi t}{L_0}\right) - \cos\left(\frac{(q-n)\pi t}{L_0}\right)\right]\right\}\Bigg) \\ &+ g(t,\tau) + O(\epsilon^2),\end{aligned} \tag{17}$$

where the bare parameters $a$, $A_n$, and $B_n$ have been replaced by their renormalized counterparts $a(\tau)$, $A_n(\tau)$, and $B_n(\tau)$. Here $g(t,\tau)$ denotes the function $g(t)$ of Eq. (15) with the same replacement. Note that then $g(t,\tau)$ is no longer a periodic function.

The RG equation $(\partial R/\partial\tau)_t=0$ consists in the present case of three independent equations

$$\frac{\partial a(\tau)}{\partial\tau}=\frac{1}{L_0}+O(\epsilon^2), \tag{18}$$

$$\frac{\partial A_n(\tau)}{\partial\tau}=\epsilon\frac{(-1)^{q+1}}{L_0}\left[\delta_{nq}+\frac{\pi}{2}\{|n-q|A_{|n-q|}-(n+q)A_{n+q}\}\right]+O(\epsilon^2), \tag{19}$$

$$\frac{\partial B_n(\tau)}{\partial\tau}=\epsilon\frac{\pi(-1)^{q+1}}{2L_0}[|n-q|B_{|n-q|}+(n+q)B_{n+q}]+O(\epsilon^2), \tag{20}$$

where we recall that the index $n$ is a positive integer. The solution to Eq. (18) is trivial: $a(\tau)=\tau/L_0+\kappa$, with $\kappa$ a constant to be determined. If one writes $A_n=\tilde{A}_n-\tilde{A}_{-n}$ and $B_n=\tilde{B}_n-\tilde{B}_{-n}$ $(n\geqslant 1)$, where the new variables satisfy

$$\frac{\partial\tilde{A}_m}{\partial\tau^\star}=\frac{2}{\pi}\delta_{mq}+(m-q)\tilde{A}_{m-q}-(m+q)\tilde{A}_{m+q}+O(\epsilon^2), \tag{21}$$

$$\frac{\partial\tilde{B}_m}{\partial\tau^\star}=(m-q)\tilde{B}_{m-q}+(m+q)\tilde{B}_{m+q}+O(\epsilon^2), \tag{22}$$

then $A_n$ and $B_n$ satisfy Eqs. (19) and (20), respectively. Here we have introduced a new time $\tau^\star\equiv\tau\epsilon\pi(-1)^{q+1}/(2L_0)$. Since this set of first order differential equations ensures the independence of the solution $R(t)$ with $\tau$, one can set $\tau=t$, which makes the terms proportional to $\tau-t$ in Eq. (17) vanish identically.

The initial conditions for these differential equations are dictated by the perturbative solution: $a(0)=\tilde{A}_m(0)=\tilde{B}_m(0)=0$. This means that $\kappa=0$ and that $\tilde{B}_m(t)=0$ for all $t$. The coefficients $\tilde{A}_m$ are not all zero due to the presence of the inhomogeneous term $(2/\pi)\delta_{mq}$. In order to solve the equation corresponding to these coefficients we introduce the generating functional $F(s,\tau^\star)=\Sigma_m s^m\tilde{A}_m(\tau^\star)$. Using Eq. (21) we see that it satisfies the following differential equation:

$$\frac{\partial F}{\partial\tau^\star}=\frac{2}{\pi}s^q+\frac{\partial F}{\partial s}[s^{q+1}-s^{1-q}], \tag{23}$$

with boundary condition $F(s,\tau^\star=0)=0$. We make the following ansatz for the solution $F(s,\tau^\star)=\Phi[e^{-\tau^\star}g(s)]+h(s)$, where $\Phi[\cdots]$, $g(s)$ and $h(s)$ are functions to be determined. Introducing this form of the generating functional into the differential equation, one determines the last two functions. The function $\Phi$ is determined once the initial boundary condition is imposed. Finally the solution reads

$$F(s,\tau^\star)=-\frac{2}{\pi q}\ln\left[\frac{e^{-q\tau^\star}(1+s^q)+e^{q\tau^\star}(1-s^q)}{2}\right]. \tag{24}$$

In order to get the coefficients $\tilde{A}_m$ we expand this solution in powers of $s$. In this way we obtain that the only nonvanishing coefficients are $\tilde{A}_{m=0}(\tau^\star)=-(2/\pi q)\ln(\cosh q\tau^\star)$ and $\tilde{A}_{m=qj}=(2/\pi qj)\tanh^j(q\tau^\star)$ with $j\in\mathbb{N}$. Note in particular that $\tilde{A}_{m<0}=0$, which then means that the original coefficients $A_n$ are equal to the $\tilde{A}_n$'s.

The RG-improved solution for $R(t)$ can be obtained from Eq. (17) by setting $\tau=t$. It is given by

$$R(t)=\frac{t}{L_0}+\sum_{j\geqslant 1}A_{qj}(t)\sin\left(\frac{qj\pi t}{L_0}\right)+\epsilon g(t,t). \tag{25}$$

Using the explicit form of the coefficients $\tilde{A}_m$ we find

$$\sum_{j\geqslant 1}A_{qj}(t)\sin\left(\frac{qj\pi t}{L_0}\right)=-\frac{2}{\pi q}\,\mathrm{Im}\,\ln[1+\xi+(1-\xi)e^{iq\pi t/L_0}], \tag{26}$$

where we have defined $\xi=\exp[(-1)^{q+1}\pi q\epsilon t/L_0]$. The (now nonperiodic) function $g(t,t)$ can be easily evaluated,

$$\begin{aligned}g(t,t)&=(-1)^q\frac{z}{L_0}\sin\left(\frac{q\pi z}{L_0}\right)\left[1+\sum_{j\geqslant 1}A_{qj}(t)qj\cos\left(\frac{qj\pi z}{L_0}\right)\right]\\&=(-1)^q\frac{z}{L_0}\sin\left(\frac{q\pi z}{L_0}\right)\\&\quad\times\left[\frac{2\xi}{1+\xi^2+(1-\xi^2)\cos(\pi qz/L_0)}\right].\end{aligned} \tag{27}$$

Finally, the RG-improved solution reads

$$\begin{aligned}R(t)=&\frac{t}{L_0}-\frac{2}{\pi q}\,\mathrm{Im}\,\ln[1+\xi+(1-\xi)e^{iq\pi t/L_0}]\\&+\epsilon(-1)^q\frac{z}{L_0}\sin\left(\frac{q\pi z}{L_0}\right)\\&\times\left[\frac{2\xi}{1+\xi^2+(1-\xi^2)\cos(\pi qz/L_0)}\right].\end{aligned} \tag{28}$$

It is worth mentioning that this solution is valid as long as $\epsilon^2t/L_0<1$, that is, the range of validity of the solution is longer than the perturbative one $(\epsilon\, t/L_0<1)$. In Fig. 1 we plot this function for the particular case $q=4$.

A solution to Eq. (3) in the long time limit was already obtained in [8] using a different procedure. It coincides with our first two terms in Eq. (28). There is perfect agreement between both solutions at long times because it can be shown that the third term in Eq. (28) is negligible in this limit (see Appendix A). However, as we have mentioned in the preceding section, this term [that comes from the periodic function $g(t)$] is crucial for the solution to satisfy the correct boundary condition at short times.

### V. ENERGY DENSITY INSIDE THE CAVITY

In order to study the local properties of the electromagnetic field inside the cavity, we concentrate on the energy density of the field

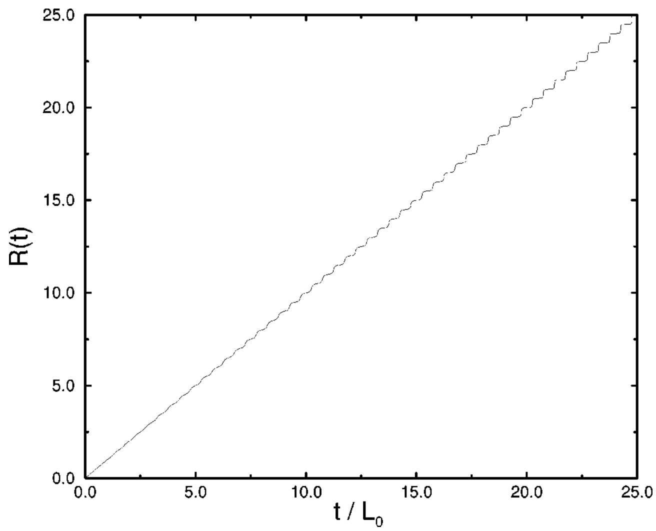


FIG. 1. $R(t)$ vs $t/L_0$ as given by Eq. (28). The values of the parameters are $q=4$ and $\epsilon=0.01$.

$$\langle T_{00}(x,t)\rangle=\frac{1}{2}\left[\left\langle\left(\frac{\partial A(x,t)}{\partial t}\right)^2\right\rangle+\left\langle\left(\frac{\partial A(x,t)}{\partial x}\right)^2\right\rangle\right], \quad (29)$$

where the expectation values are taken with respect to the vacuum state. Using the well-known point splitting method to regularize the divergence appearing in the energy density [3], one can obtain the following expression for the renormalized energy density, $\langle T_{00}(x,t)\rangle=-f(t+x)-f(t-x)$, where

$$f=\frac{1}{24\pi}\left[\frac{R'''}{R'}-\frac{3}{2}\left(\frac{R''}{R'}\right)^2+\frac{\pi^2}{2}(R')^2\right]. \quad (30)$$

This expression involves second and third derivatives of $R(t)$. As $R'(t)$ is discontinuous at $t=(2p+1)L_0$, $p=0,1,2,\ldots$ [see Eq. (28)], then the energy density will develop a $\delta$ function singularity which will be infinitely reflected back and forth between the mirrors. The physical origin of this singularity is the initial discontinuity of the wall velocity. We will ignore this singularity in what follows.

The structure of the electromagnetic field within the cavity for our solution $R(t)$ is similar to that for other existing solutions in the literature. In particular, for $q\geqslant 2$ the energy density grows exponentially in the form of $q$ traveling wave packets which become narrower and higher as time increases. The total energy within the plates increases exponentially at the expense of the energy needed to keep the plate moving. In Fig. 2 we show the energy density profile between plates for a fixed time and for the case $q=4$. As time evolves, the peaks move back and forth bouncing against the mirrors. The height of the peaks increases as $e^{2\pi q\epsilon t/L_0}$ and their width decreases as $e^{-\pi q\epsilon t/L_0}$, so that the total area beneath each peak, and hence the total energy, grows as $e^{\pi q\epsilon t/L_0}$. Apart from this exponential growth, there are "sub-Casimir" regions: between the peaks the energy density takes values $q^2$ times smaller than the static Casimir case, $\langle T_{00}\rangle_{\text{static}}=-\pi/24L_0^2$. One can prove all these properties analytically by computing the energy density with the solution given in Eq. (28) and its derivatives (see Appendix B). In Fig. 3 the energy density is shown as a function of time at the midpoint between plates, also for the $q=4$ case.

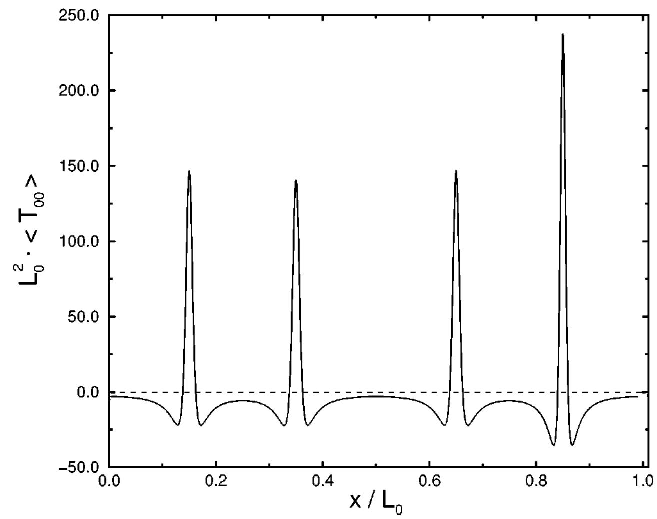


FIG. 2. Energy density profile between plates for fixed time $t/L_0=20.4$ for the $q=4$ case. The amplitude coefficient is $\epsilon=0.01$.

A rather different picture appears when one considers the $q=1$ case, that corresponds to an oscillation frequency equal to the fundamental frequency of the cavity. In this case the energy density at a given point oscillates in time around the static Casimir value, and its time average coincides with that value.

## VI. CONCLUSIONS

In this paper we have studied the one-dimensional dynamical Casimir effect of a resonant oscillating cavity. For this one-dimensional case, the modes of the electromagnetic field can be expressed in terms of the solution to the so-called Moore equation. We have used a renormalization-group improvement of the naive perturbative solution and we have succeeded in obtaining an analytic solution which is

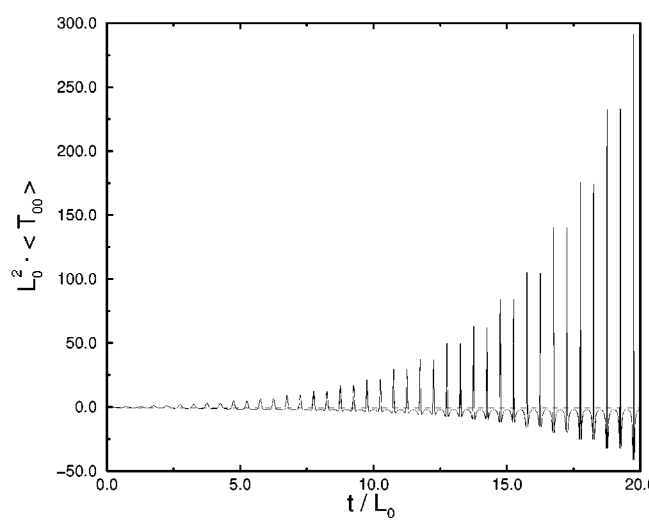


FIG. 3. Energy density as a function of time for the midpoint $x/L_0=0.5$ between plates. The parameters are $q=4$ and $\epsilon=0.01$.

valid up to times $t<L_0\epsilon^{-2}$, thus extending the range of validity of the perturbative solution ($t<L_0\epsilon^{-1}$). We have calculated the energy density inside the cavity and we have shown that a nontrivial structure appears, with a series of peaks that grow exponentially in time and move back and forth bouncing against the mirrors. Although this structure has already been found in a previous work [6], here we have presented an analytic derivation based on the renormalization-group method described in [10]. We expect this method to be useful to analyze the more realistic situation of a three-dimensional oscillating cavity. This analysis can be performed by studying the set of differential equations satisfied by the modes of the electromagnetic field [9]. This topic will be the subject of further investigation.

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## APPENDIX A: SHORT TIME AND LONG TIME BEHAVIOR OF $R(t)$

In this appendix we analyze the short time ($\epsilon\, t/L_0 \ll 1$) and long time ($\epsilon\, t/L_0 \gg 1$) behavior of the RG-improved function $R(t)$ given in Eq. (28). Let us first split the solution as $R(t)=R_s(t)+R_{\rm np}(t)$, where

$$R_s(t)=\frac{t}{L_0}-\frac{2}{\pi q}\,\mathrm{Im}\,\ln[1+\xi+(1-\xi)e^{iq\pi t/L_0}]$$
$$=\frac{t}{L_0}-\frac{2}{\pi q}\arctan\left[\frac{\sin(q\pi t/L_0)}{(1+\xi)/(1-\xi)+\cos(q\pi t/L_0)}\right], \tag{A1}$$

$$R_{\rm np}(t)=\epsilon(-1)^q\frac{z}{L_0}\sin\left(\frac{q\pi z}{L_0}\right)\times\left[\frac{2\xi}{1+\xi^2+(1-\xi^2)\cos(\pi q z/L_0)}\right], \tag{A2}$$

with

$$\xi=\exp\left[\frac{(-1)^{q+1}\pi q\epsilon t}{L_0}\right]. \tag{A3}$$

The function $R_{\rm np}$ stems from the RG improvement of the periodic function $g(t)$, and it is nonperiodic. The variable $z$ ($-L_0\leqslant z\leqslant L_0$) is given in terms of $t$ as $z=t-2pL_0$ with $p=0,1,2,\ldots$ . This integer $p$ is obtained from the value that $t$ takes as $p=\frac{1}{2}\,\mathrm{int}(t/L_0)$ or $p=\frac{1}{2}[\mathrm{int}(t/L_0)+1]$, for $\mathrm{int}(t/L_0)$ even or odd, respectively.

For the short time limit $t\ll\epsilon^{-1}L_0$, these functions are

$$R_s(t)\approx\frac{t}{L_0}-(-1)^q\frac{\epsilon t}{L_0}\sin\left(\frac{q\pi t}{L_0}\right),$$

$$R_{\rm np}(t)\approx(-1)^q\frac{\epsilon z}{L_0}\sin\left(\frac{q\pi z}{L_0}\right), \tag{A4}$$

which then leads to the perturbative solution given in Eq. (16).

For the long time limit $t\gg\epsilon^{-1}L_0$ (but $t<\epsilon^{-2}L_0$ since this poses the upper limit for the validity of our RG solution), we analyze $R_s$ and $R_{\rm np}$ separately. We want to show that in this limit, the latter function is negligible. This can be graphically verified, but here we present an analytical demonstration. The function $R_s$ has a first term, linear in time, and a second one, that for late times becomes an oscillating function. The amplitude of the oscillations is independent of $\epsilon$. Due to this second term, $R_s$ develops a staircase form for long times, as shown in Fig. 1. Within regions of $t$ between odd multiples of $L_0$ (i.e., in each period $p$), there appear $q$ jumps, located at values of $t$ satisfying $\cos(q\pi t/L_0)=\mp 1$, the upper sign corresponding to even values of $q$ and the lower one to odd values of $q$. Next we calculate the first derivative of $R_s$. Since $d\xi/dt$ is proportional to $\epsilon\xi$, one can differentiate the function $R_s$ with respect to time, treating $\xi$ as a constant. The first derivative is then $R_s'(t)=2\xi\psi(t)$, where

$$\psi(t)=\frac{1}{L_0[1+\xi^2+(1-\xi^2)\cos(q\pi t/L_0)]}. \tag{A5}$$

Using Eq. (A3), we see that $\xi$ vanishes (diverges) exponentially for $q$ even (odd) at long times. For even $q$, the first derivative $R_s'$ develops peaks for times $t_n=[(2n+1)L_0]/q$, with $n$ an integer. The height of these peaks grows exponentially as $\xi^{-1}$. Between peaks, $R_s'$ vanishes exponentially for long times. In a similar fashion, for odd $q$, the first derivative develops peaks at $t_n=2nL_0/q$. In the following, we shall consider only the even case, the odd one being completely similar.

Let us now analyze the function $R_{\rm np}$. Once again treating $\xi$ as a constant when differentiating with respect to time, this function can be expressed in terms of the first derivative of $R_s$ as follows:

$$R_{\rm np}(t)=(-1)^q\epsilon z\sin\left(\frac{q\pi t}{L_0}\right)R_s'(t). \tag{A6}$$

We see that $R_{\rm np}$ is the product of a bounded factor ($|z|<L_0$) times a function $F(t)\equiv\sin(q\pi t/L_0)R_s'(t)$ that is proportional to the first derivative of $R_s$ and might thus be unbounded. We shall now show that this is *not* the case. Far from the position of the peaks, $F$ is bounded because $R_s'$ is. In a surrounding of $t_n$, we express $F$ as

$$F(\delta)=-\frac{2\pi q\xi\delta}{L_0[(1-\xi^2)(\pi^2q^2\delta^2/2)+2\xi^2]}, \tag{A7}$$

where $\delta=(t-t_n)/L_0$. First, we note that this function vanishes for $\delta=0$, i.e., $R_{\rm np}(t=t_n)=0$. Second, this function has extrema equidistant from $t_n$ located at $\delta_\pm=\pm 2\xi/\pi q\sqrt{1-\xi^2}$ and at these points $F(\delta_\pm)=\mp 1/L_0\sqrt{1-\xi^2}$. Since for long times $\xi\to 0$, we conclude that $F(\delta_\pm)$ is bounded by $1/L_0$. Consequently, $R_{\rm np}$ is a correction of order $\epsilon$ to the second term of Eq. (A1).

## APPENDIX B: STRUCTURE OF THE ELECTROMAGNETIC FIELD

In this appendix we study briefly the structure of the electromagnetic field within the cavity as given by Eqs. (29) and (30) for our solution $R(t)$, in order to understand the form of the energy density profile shown in Figs. 2 and 3 in the long time regime $\epsilon t/L_0 > 1$. As in Appendix A we will split $R(t)$ as $R(t) = R_s + R_{\rm np}$ [see Eqs. (A1) and (A2)]. In order to analyze the energy-momentum tensor, we need to study the first three derivatives of the solution $R(t)$. For $R_s$ we get

$$R_s'(t) = 2\xi\psi(t), \tag{B1}$$

$$R_s''(t) = 2\xi(1-\xi^2)\pi q \sin\left(\frac{\pi q t}{L_0}\right)\psi^2(t), \tag{B2}$$

$$R_s'''(t) = 2\xi(1-\xi^2)(\pi q)^2\left\{(1+\psi^2)\cos\left(\frac{\pi q t}{L_0}\right) + (1-\xi^2)\left[1+\sin^2\left(\frac{\pi q t}{L_0}\right)\right]\right\}\psi^3(t), \tag{B3}$$

$$\psi(t) = \frac{1}{L_0[1+\xi^2+(1-\xi^2)\cos(q\pi t/L_0)]}, \tag{B4}$$

where again we treat $\xi$ as a constant. For even $q$, all these derivatives develop peaks at times $t_n = [(2n+1)L_0]/q$. As can be easily seen from the above equations, the height of the peaks for the $m$th derivative of $R_s$ is proportional to $\xi^{-m}$. Using the same methods as in Appendix A, one can show (after some algebra) that near the times $t_n$, the $m$th derivative of $R_{\rm np}$ also has peaks whose heights are proportional to $\epsilon\xi^{-m}$. Since $\epsilon \ll 1$, it means that at long times all the derivatives of $R_{\rm np}$ are negligible with respect to those of $R_s$. The form of the energy-momentum tensor will be governed just by the first part of our solution, namely, by $R_s$.

Let us concentrate only on the contribution to the energy density which is proportional to $R'^2$ [see Eq. (30)]. From the above discussion it is clear that $T_{00}$ will develop peaks which grow as $e^{2\pi q\epsilon t/L_0}$. Their width decreases exponentially as $e^{-\pi q\epsilon t/L_0}$, so the total area of the peaks grows exponentially. The same holds for the total energy in the cavity. The analysis of the other two terms of Eq. (30) leads to the same conclusion.

The case $q=1$ shows a different behavior. Indeed, when the energy density of the field is computed using the derivatives of $R_s$ given above, there is a cancellation between the different contributions in Eq. (30) and the final answer coincides with the static Casimir value.


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